Differentiated-Products Cournot Attributes Higher Markups Than Bertrand-Nash

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July 2022

Abstract

In a differentiated products setting when costs are unobserved, the Cournot model of quantity-setting competition attributes a greater share of prices to markups than does the Bertrand-Nash model of price-setting, leading to lower estimates of marginal costs.

JEL classification: L1, *Keywords:* Cournot competition, Bertrand competition, markups

1 Introduction

The classic price-setting model of Bertrand and quantity-setting model of Cournot are the first models of oligopolistic competition every student learns. In the simplest case of homogenous goods and symmetric and constant marginal costs, the former makes the stark prediction of zero markups, while the latter allows for positive markups and firm profits. The Bertrand-Nash model of price setting in differentiated products markets is the basis of an enormous amount of empirical work in industrial organization; the quantity-setting model is also well-defined in such settings, and while less popular, is also considered empirically. We show that in a general differentiated products environment where all products are gross substitutes and there are no Giffen goods, the Cournot model infers higher markups from the same observables, attributing a smaller fraction of prices to marginal costs. Our result implies that in structural empirical settings, researchers choosing to model conduct as a Cournot game rather than a Bertrand game will mechanically measure a higher level of markups.

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2 Model and Result

2.1 Environment

Let $\mathcal{N} = \{1, 2, \ldots, N\}$ be the set of products and $\mathcal{F}$ a partition of $\mathcal{N}$ the set of firms, so that $f \in \mathcal{F}$ refers both to the identity of a firm and to the set of products it sells. Let $p_1, \ldots, p_N$ denote prices and $s_1, \ldots, s_N$ market shares. Normalize market size to 1, so demand and market share for a product are identical. Suppose the demand system $s(p)$ is known.

Assumption 1. The demand system $s(\cdot)$ is differentiable with respect to prices, and for each product $i \in \mathcal{N}$, (i) $\frac{\partial s_i}{\partial p_i} < 0$, (ii) $\frac{\partial s_j}{\partial p_i} > 0$ for each $j \neq i$, and (iii) $\frac{\partial}{\partial p_i} \sum_{j \in \mathcal{N}} s_j < 0$.

That is, in addition to differentiability, we assume that there are no Giffen goods, goods are gross substitutes, and the outside good is a gross substitute for any of the other goods. All of these conditions occur naturally in any discrete choice setting where consumers choose whichever single product (or the outside option) maximizes their utility, given a utility from each product that is decreasing in that product's price (and independent of other products' prices). In particular, Assumption 1 holds in all the standard demand systems used in empirical industrial organization, including that of Berry, Levinsohn, and Pakes (1995). Outside of discrete choice settings, these are of course substantive assumptions, particularly the requirement that no two goods are complements.[1]

Assumption 2. Marginal costs are constant for each product (but need not be equal across products).

Let c_j denote the marginal cost of good j. Let p, c, and s be vectors of all products' prices, marginal costs, and market shares, respectively. Let S_p be the Jacobian of the demand system with respect to prices, $[S_p]_{i,j} = \frac{\partial s_i}{\partial p_j}$. Under Assumption 1, S_p is invertible, so $s(\cdot)$ is invertible as well.

2.2 Bertrand-Nash (Price-Setting) Competition

In the Bertrand-Nash model, each firm f sets prices for all its products to solve

$$\max_{\{p_j\}_{j \in f}} \sum_{j \in f} (p_j - c_j) s_j(p)$$

[1] Product substitutability is also key to establishing the ranking of equilibrium prices between the Cournot and Bertrand models given a fixed level of marginal costs. Okuguchi (1987) and Amir and Jin (2001) show that Cournot prices may be lower than Bertrand prices when some products are complements.

taking as given the prices of products $i \notin f$. The firms' first-order conditions with respect to each price can be stacked in matrix form and written as

$$p - c \quad = \quad -\left(\Omega \odot S_p'\right)^{-1} s$$

where Ω the ownership matrix (defined by $\Omega_{i,j} = 1$ if i and j are products sold by the same firm and 0 otherwise) and $\odot$ refers to element-by-element multiplication of matrices ($[A \odot B]_{i,j} = A_{i,j}B_{i,j}$). Thus, under the assumption that firms are playing the equilibrium of the Bertrand game, markups can be recovered from the observed market shares, price elasticities at equilibrium prices, and the ownership matrix; if prices are also observed, marginal costs can be recovered as prices minus markups.

2.3 Cournot (Quantity-Setting) Competition

In the Cournot model, each firm f instead chooses quantity levels s_j for each of its products, taking as fixed the quantity levels of competing products, solving

$$\max_{\{s_j\}_{j\in f}} \sum_{j\in f} (p_j(s) - c_j)s_j$$

where $p_j(s) = s_j^{-1}(s)$ is the price of product j given the market shares chosen by all firms. Since $\left[\frac{\partial s^{-1}}{\partial s}\right] = S_p^{-1}$, the first-order conditions can be stacked and written as

$$p - c \quad = \quad -\left(\Omega \odot \left(S_p^{-1}\right)'\right) s$$

so once again markups and marginal costs can be recovered under the assumption firms are playing the equilibrium of the Cournot game.

2.4 Result

Rather than fixing primitives and considering the different equilibrium outcomes of the two models, our thought experiment is the reverse: a researcher knows the demand system and observes market outcomes but not costs, and infers markups and costs from the equilibrium implications of one of the two models. Our result is the following:

Theorem 1. *Fixing the demand system and observable outcomes (prices and market shares), the Cournot model implies higher markups and therefore lower marginal costs than the Bertrand model.*

For a simple example to better visualize the result, suppose there are four products, two firms $f_1 = \{1, 2\}$ and $f_2 = \{3, 4\}$ selling two products each, and that $\frac{\partial s_i}{\partial p_j} = -0.45$ if $j = i$ and 0.05 otherwise.[2] In that case, the Bertrand model implies markups are

$$p - c \quad = \quad - \begin{bmatrix} -0.45 & 0.05 & 0 & 0 \\ 0.05 & -0.45 & 0 & 0 \\ 0 & 0 & -0.45 & 0.05 \\ 0 & 0 & 0.05 & -0.45 \end{bmatrix}^{-1} s \quad = \begin{bmatrix} 2.25 & 0.25 & 0 & 0 \\ 0.25 & 2.25 & 0 & 0 \\ 0 & 0 & 2.25 & 0.25 \\ 0 & 0 & 0.25 & 2.25 \end{bmatrix} s$$

while the Cournot model implies markups of

$$p - c \quad = \quad - \left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \odot [S_p']^{-1} \right) s \quad \approx \quad \begin{bmatrix} 2.33 & 0.33 & 0 & 0 \\ 0.33 & 2.33 & 0 & 0 \\ 0 & 0 & 2.33 & 0.33 \\ 0 & 0 & 0.33 & 2.33 \end{bmatrix} s$$

Thus, for any vector of market shares, the Cournot markups are higher. The theorem shows the result holds generally, for the same reason. If we (without loss of generality) relabel products so that each firm sells products with consecutive indices, so that Ω, $-(\Omega \odot S_p')^{-1}$, and $-(\Omega \odot (S_p^{-1})')$ are all block-diagonal matrices, we establish that the within-block elements of the latter two matrices are all strictly positive, with the elements of $-(\Omega \odot (S_p^{-1})')$ each being strictly larger. Since $s > 0$, this implies larger markups under the Cournot model.

3 Discussion

A number of papers consider the reverse of our question: for a given set of primitives (demand system and marginal costs), how do equilibrium outcomes compare across the two models? Vives (1985) studies a demand system very similar to ours, derived from a representative consumer with quasilinear preferences but with single-product firms and symmetric cross-price effects $\frac{\partial s_i}{\partial p_j} = \frac{\partial s_j}{\partial p_i} > 0$. Under additional assumptions that guarantee *either* (i) symmetric demand and uniqueness of equilibrium under both models, or (ii) quasi-concave profit functions and Bertrand reaction functions that are increasing in rival prices, he shows that the Cournot model gives higher equilibrium prices than the Bertrand model.[3] Related

[2] These arise with simple logit demand $s_i = \frac{\exp(v_i - \alpha p_i)}{1 + \sum_{j=1}^{4} \exp(v_j - \alpha p_j)}$ if $s_1 = s_2 = s_3 = s_4 = 0.10$ and $\alpha = 5$.

[3] Formally, in the latter case, the result is that for any equilibrium of the Cournot model, there is an equilibrium of the Bertrand model with lower prices.

results appear in Hathaway and Rickard (1979), Singh and Vives (1984), Cheng (1985), and Okuguchi (1987). Because our result concerns markups and costs implied by each model given equilibrium outcomes, rather than predictions on equilibrium outcomes given primitives, we can avoid imposing additional assumptions to ensure equilibrium uniqueness, a key aspect of the previous literature.

Note also that our result does not depend on constant marginal costs; if the total cost $C_j(s_j)$ of producing each product j depended nonlinearly on the quantity produced, the first-order conditions would be identical, with $C'_j(s_j)$ replacing c_j. The Cournot model would infer greater markups (lower marginal costs) for each product at the observed demand level. If firms had economies of scope, however – the marginal cost of product j depending on the quantity produced of the firm's *other* products – the first-order conditions would change and our result need not hold.

4 Acknowledgements

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

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Appendix – Proof

The Bertrand-Nash model gives first-order conditions

$$s_i + \sum_{j \in f} (p_j - c_j) \frac{\partial s_j}{\partial p_i} \quad = \quad 0$$

Stacking these in matrix form gives

$$\left(\Omega \odot \begin{bmatrix} \frac{\partial s_1}{\partial p_1} & \frac{\partial s_2}{\partial p_1} & \cdots & \frac{\partial s_N}{\partial p_1} \\ \frac{\partial s_1}{\partial p_2} & \frac{\partial s_2}{\partial p_2} & \cdots & \frac{\partial s_N}{\partial p_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial s_1}{\partial p_N} & \frac{\partial s_2}{\partial p_N} & \cdots & \frac{\partial s_N}{\partial p_N} \end{bmatrix} \right) \begin{bmatrix} p_1 - c_1 \\ p_2 - c_2 \\ \vdots \\ p_N - c_N \end{bmatrix} \quad = \quad - \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_N \end{bmatrix}$$

Under Assumption 1, $\Omega \odot S_p'$ is invertible, giving

$$p - c \quad = \quad -\left(\Omega \odot S_p'\right)^{-1} s$$

The Cournot model gives first-order conditions

$$p_i - c_i + \sum_{j \in f} \frac{\partial p_j}{\partial s_i} s_j \quad = \quad 0$$

or, stacking,

$$p - c \quad = \quad -\left(\Omega \odot \left[\frac{\partial p}{\partial s}\right]'\right) s$$

Since $p(\cdot) = s^{-1}$, $\left[\frac{\partial p}{\partial s}\right] = S_p^{-1}$, and therefore the Cournot model gives markups

$$p - c \quad = \quad -\left(\Omega \odot \left(S_p^{-1}\right)'\right) s$$

Note that the two models give very similar-looking markups, the difference being that for Cournot markups, we invert S_p' before multiplying by Ω and for Bertrand markups, we multiply by Ω before inverting.

It suffices to show the result for one firm's products. Without loss, we can relabel products so that each firm sells products numbered consecutively, so that Ω is a block diagonal matrix, with firm 1 selling products 1 through m. Let $A = -S_p'$, so that markups are $(\Omega \odot A)^{-1} s$ for Bertrand and $(\Omega \odot A^{-1}) s$ for Cournot. Note that $(\Omega \odot A)^{-1}$ and $\Omega \odot A^{-1}$ will both be block diagonal matrices, with the blocks corresponding to products sold by the same firm.

We will show that for $i, j \in f$ (two products sold by the same firm and therefore nonzero elements of these matrices), $[\Omega \odot A^{-1}]_{i,j} > [(\Omega \odot A)^{-1}]_{i,j} > 0$, from which (since $s > 0$) the result follows.

We begin by establishing a decomposition of the matrix $A = -S_p^{-1}$.

Claim 1. *Under Assumption 1, the matrix A can be written as a matrix product $A = LU$, where L is an $N \times N$ lower-triangular matrix with positive diagonal terms and negative below-the-diagonal terms and U is an $N \times N$ upper-triangular matrix with positive diagonal terms and negative above-the-diagonal terms.*

Theorem 2.5.3 in Horn and Johnson (1994) (pp. 114-115) establishes that A is an M-matrix. ($A \in Z_n$, since it has negative off-diagonal elements. $A' = -S_p$ is strictly row diagonally dominant and has positive diagonal elements, so (by 2.5.3.13) A' is an M-matrix, so A is as well.) It then follows (by 2.5.3.9) that it can be written as a matrix product $A = LU$, where L is lower-triangular, U is upper-triangular, and both have strictly positive diagonal elements.

That the below-diagonal terms of L and the above-diagonal terms of U are negative requires a separate proof (and in fact is the second Exercise on page 117 of Horn and Johnson (1994)). Suppose this were not true. List the off-diagonal terms in the following order:

$$(z_n) \quad = \quad (L_{2,1},\ U_{1,2},\ L_{3,1},\ L_{3,2},\ U_{1,3},\ U_{2,3},\ L_{4,1},\ L_{4,2},\ L_{4,3},\ U_{1,4},\ U_{2,4},\ U_{3,4},\ L_{5,1}, \ldots)$$

(That is, begin with the second row of L, then the second column of U, then the third row of L, the third column of U, and so on, omitting the diagonal and above-diagonal elements of L and the diagonal and below-diagonal elements of U.) Letting sign superscripts denote matrix elements whose sign is already known, note that since $A^{(-)}_{2,1} = L_{2,1} U^{(+)}_{1,1}$, we know that $L_{2,1} < 0$, and similarly $U_{1,2} < 0$ because $A^{(-)}_{1,2} = L^{(+)}_{1,1} U_{1,2}$.

Find the first element in the list z_n which is non-negative. First, suppose it is an element of L. This means that for some (i, j) with $i > j$, $L_{i,j} \geq 0$, but $L_{i,j'} < 0$ for $j' < j$, $L_{i',j'} < 0$ for $i' < i$, and $U_{i',j'} < 0$ for $j' < i$. Note that

$$A^{(-)}_{i,j} \quad = \quad \sum_{k=1}^{N} L_{i,k} U_{k,j} \quad = \quad \sum_{k=1}^{j} L_{i,k} U_{k,j} \quad = \quad \sum_{k=1}^{j-1} L^{(-)}_{i,k} U^{(-)}_{k,j} + L_{i,j} U^{(+)}_{j,j}$$

requiring $L_{i,j} < 0$, a contradiction. Similarly, if the first nonnegative element of z_n is an element of U, $U_{i,j} \geq 0$ while $U_{i',j} < 0$ for $i' < i$, $U_{i',j'} < 0$ for $j' < j$, and $L_{i',j'} < 0$ for $i' \leq j$.

This time, $i < j$, so

$$A_{i,j}^{(-)} \quad = \quad \sum_{k=1}^{N} L_{i,k} U_{k,j} \quad = \quad \sum_{k=1}^{i} L_{i,k} U_{k,j} \quad = \quad \sum_{k=1}^{i-1} L_{i,k}^{(-)} U_{k,j}^{(-)} + L_{i,i}^{(+)} U_{i,j}$$

requiring $U_{i,j} < 0$, a contradiction. Thus, there can be no first nonnegative element of the list (z_n), so all below-diagonal elements of L, and all above-diagonal elements of U, are strictly negative.

Claim 2. *Under Assumption 1, $A = LU$, L^{-1} is a lower-triangular matrix with all diagonal and below-diagonal elements strictly positive, and U^{-1} is an upper-triangular matrix with all diagonal and above-diagonal elements strictly positive.*

Let $R = L^{-1}$, so that

$$\begin{bmatrix} L_{1,1} & & & \\ L_{2,1} & L_{2,2} & & \\ \vdots & \vdots & \ddots & \\ L_{N,1} & L_{N,2} & \cdots & L_{N,N} \end{bmatrix} \begin{bmatrix} R_{1,1} & R_{1,2} & \cdots & R_{1,N} \\ R_{2,1} & R_{2,2} & \cdots & R_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N,1} & R_{N,2} & \cdots & R_{N,N} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Note that $R_{1,1} = 1/L_{1,1}$ and $R_{1,k} = 0$ for $k > 1$; then $R_{2,2} = 1/L_{2,2}$ and $R_{2,k} = 0$ for $k > 2$, and so on, so R is lower triangular with strictly positive diagonal terms,

$$\begin{bmatrix} L_{1,1}^{(+)} & & & \\ L_{2,1}^{(-)} & L_{2,2}^{(+)} & & \\ \vdots & \vdots & \ddots & \\ L_{N,1}^{(-)} & L_{N,2}^{(-)} & \cdots & L_{N,N}^{(+)} \end{bmatrix} \begin{bmatrix} R_{1,1}^{(+)} & & & \\ R_{2,1} & R_{2,2}^{(+)} & & \\ \vdots & \vdots & \ddots & \\ R_{N,1} & R_{N,2} & \cdots & R_{N,N}^{(+)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Suppose there was some i, j with $i > j$ such that $R_{i,j} \leq 0$, and assume without loss that (i, j) is the first such occurrence (again listing elements by row). Then

$$0 \quad = \quad I_{i,j} \quad = \quad \sum_{k=1}^{N} L_{i,k} R_{k,j} \quad = \quad \sum_{k=j}^{i} L_{i,k} R_{k,j} \quad = \quad \sum_{k=j}^{i-1} L_{i,k}^{(-)} R_{k,j}^{(+)} + L_{i,i}^{(+)} R_{i,j}$$

(The third equality is because $R_{k,j} = 0$ for $k < j$ and $L_{i,k} = 0$ for $k > i$.) If $R_{i,j} \leq 0$, then the right-hand side would be strictly negative; so $R_{i,j} > 0$, giving a contradiction. So L^{-1} is lower-triangular with all strictly positive diagonal and below-diagonal terms; the analogous argument establishes U^{-1} upper-triangular with strictly positive diagonal and above-diagonal terms.

Claim 3. *For $i, j \leq |f|$, $(\Omega \odot A^{-1})_{i,j} > (\Omega \odot A)^{-1}_{i,j} > 0$.*

Let $m = |f|$, so firm 1 sells the first m products. Since $A = LU$, $A^{-1} = U^{-1}L^{-1}$. We established above that $U^{-1}_{i,k}$ is 0 for $k < i$ and strictly positive for $k \geq i$, and $L^{-1}_{k,j}$ is 0 for $k < j$ and strictly positive for $k \geq j$, so

$$A^{-1}_{i,j} = \sum_{k=1}^{N} U^{-1}_{i,k} L^{-1}_{k,j} = \sum_{k=\max\{i,j\}}^{N} U^{-1}_{i,k} L^{-1}_{k,j}$$

where each term in the latter sum is strictly positive. For $i, j \leq m$, $\Omega_{i,j} = 1$ and therefore

$$(\Omega \odot A^{-1})_{i,j} = \sum_{k=\max\{i,j\}}^{N} U^{-1}_{i,k} L^{-1}_{k,j}$$

For any matrix M, let $\widehat{M}$ denote the $m \times m$ leading principal submatrix of M, i.e., the submatrix consisting of the first m rows and columns of M. Since block-diagonal matrices invert block by block, $\widehat{(\Omega \odot A)}^{-1} = (\widehat{A})^{-1}$. It's easy to establish that $\widehat{A} = \widehat{L}\widehat{U}$ and therefore $\widehat{A}^{-1} = \widehat{U}^{-1}\widehat{L}^{-1}$. As a result, for $i, j \leq m$,

$$(\Omega \odot A)^{-1}_{i,j} = \sum_{k=1}^{m} U^{-1}_{i,k} L^{-1}_{k,j} = \sum_{k=\max\{i,j\}}^{m} U^{-1}_{i,k} L^{-1}_{k,j} < \sum_{k=\max\{i,j\}}^{N} U^{-1}_{i,k} L^{-1}_{k,j} = (\Omega \odot A^{-1})_{i,j}$$

giving the result. □